Some Applications of Fractional Calculus to a Certain Class Defined by Ruscheweyh Derivative

H. M. Hossen, N. Asker, F. H. Majrapi and B. M. Al-Awad
Department of Mathematic, Faculty of Science
Jazan Universty, Jazan , Saudi Arabia
hhossen@iazamu.edu.sa, naskar@jazanu.edu.sa, foofe.15@icloud.com, bb321111@hotmail.com

Abstract: the main objective of this paper to find distortion inequalities involving fractional calculus for functions belonging to the class $R_n^*(\alpha)$ which studied previously by Hossen [5] in addition to applying the method used earlier by Cho and Aouf [3].

Mathematics Subject Classification: 30C45.
Keywords: Univalent function, Ruscheweyh derivative, fractional derivative of order λ, fractional integral of order λ.

I. Introduction

Geometric Function Theory interesting with the relation between the properties of functions and geometric properties of the image of unite disc under the function . A complex valued function $f(z)$ is univalent if it satisfies the condition

$$f(z)=f(w) \quad \Leftrightarrow \quad z=w, \quad z,w\in\mathbb{C} \tag{1}$$

On another hand, we say that a complex-valued function $f(z)$ is analytic if you can differentiate it infinitely many times and it can always be represented as a Taylor series as follows:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k, \qquad (a_k\geq 0,\ k\in N=\{1.2.\cdots\}), \tag{2}$$

and normalized by $f(0)=0$ and $f'(0)=1$.

Let T denote the class of functions $f(z)$ defined by (2) are analytic and univalent in the open unite disc $U=\{z:|z|<1\}$.

A function $f(z)\in T$ is called stralike function of order α (see [15]) if $f(z)$ satisfies the following condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad (0\le\alpha<1;\, z\in U)\ , \tag{3}$$

Note that $S^*(0)=S^*$ is the class of starlike functions (see [4]) .

In 1975 Ruscheweyh [14] defined the operator $D^n f(z)$ by

$$D^n f(z)=\frac{z\left(z^{n-1}f(z)\right)^{n}}{n!},\quad \left(n\in N_0=N\cup\{0\}\right).\ , \tag{4}$$

We observe that $D^0 f(z)=f(z)$ and $D^1 f(z)=zf'(z)$.

From equation (4) after simple calculation, we can write Ruscheweyh derivative by

$$D^n f(z)=z+\sum_{k=2}^{\infty}\delta(n,K)\,a_k z^k,\quad (n\in N_0,\, z\in U) \tag{5}$$

where

$$\delta(n,K)=\frac{(n+k-1)!}{n!(k-1)!}. \tag{6}$$

In 2001 Hossen [5] use the Ruscheweyh derivative $D^n f(z)$ to define the class $R_n^*(\alpha)$ which consists of all function $f(z)\in T$ satisfies the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\}>\frac{n+\alpha}{n+1},\qquad (0\le\alpha<1,\, z\in U)\cdot \tag{7}$$

By specializing the parameters n and α, in the definition of the class can be reduced to classes studied pervious by different researchers:

i. Put $n=0$, we get $R_0^*(\alpha)=T^*(\alpha)$, studied by Silverman [15];

ii. Put $\alpha=0$, we get $R_n^*(0)=R_n^*$, studied by Owa [13];

To prove the main result, we need to recall here the definition of fractional calculus which depends on gamma function so; we state the definition of gamma function (see [1]) given by

$$\Gamma(m)=\int_0^{\infty}t^{m-1}e^{-t}\,dt\ \cdot \tag{8}$$

the properties of gamma function and values (see for example [8]).

Nowadays, fractional calculus is a rich area of research in Mathematics, Physics, Chemistry and Engineering (see for example [7], [9] and [10]. There are many definition of fractional calculus (fractional integral and fractional derivative) studied by different researches (for an overview see [3], [6], [11] and [16]). Now, we recall here the definition given by Owa [12].

Definition. 1.1. The fractional integral of order λ is defined for a function $f(z)$ by

$$D_z^{-\lambda} f(z)=\frac{1}{\Gamma(\lambda)}\int_0^{z}\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta, \tag{9}$$

where $\lambda>0$, $f(z)$ is analytic function in a simple-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition1.2. The fractional derivative of order λ is defined for a function $f(z)$ by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{10}$$

where $0<\lambda<1$, $f(z)$ is constrained and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.1.

From Definitions 1.2, it is easy to see that

$$D_z^{\lambda}\{z^k\} = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda}. \tag{11}$$

Definition1.3. Under the hypotheses of Definition 1.2, the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} f(z), \tag{12}$$

where $0<\lambda<1$ and $n \in N_0$.

The problem of coefficient estimates is one of interesting problems which was studied by researchers for certain classes in the open unit disc. Closely related to this problem using the results of Hossen [5] to determine application of fractional calculus to functions belong to the class $R_n^*(\alpha)$ details with some application of computers software .

II. METHODOLOGY

Using coefficient estimates for the class $R_n^*(\alpha)$ which studied previously by Hossen [5] in addition to applying the method used earlier by Cho and Aouf [3] to obtain distortion inequalities associated with the fractional calculus for the functions $f(z) \in R_n^*(\alpha)$.

III. MAIN RESULTS

In order to prove our results, we need the following Lemma due to Hossen [5] :

Lemma 3.1: Let the function $f(z)$ defined by (2). Then $f(z) \in R_n^*(\alpha)$ if and only if

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) a_k \le 1-\alpha . \tag{13}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)\delta(n,k)} z^k, \qquad (k \ge 2). \tag{14}$$

Now, we study distortion inequalities associated with the fractional derivative of order λ for the function $f(z) \in R_n^*(\alpha)$.

Theorem 3.1. Let the function $f(z)$ given by (2) be in the class $R_n^*(\alpha)$, then, we have

$$\left|D_z^{-\lambda} f(z)\right| \le \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{2(1-\alpha)}{(2+\lambda)(2-\alpha)(n+1)} |z|\right\}, \tag{15}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{1-\alpha}{(2+\lambda)(2-\alpha)(n+1)}|z|\right\}, \tag{16}$$

for $\lambda>0$, $0\leq\alpha<1, n\in N_0$ and $z\in U$. The result is sharp for the function $f(z)$ defined by

$$D_z^{-\lambda} f(z)=\frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{1-\alpha}{(2+\lambda)(2-\alpha)(n+1)}z\right\}. \tag{17}$$

Proof. Using the technique used ealier by Cho and Aouf [3].

Let

$$F(z)=\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda} f(z)=z-\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}a_k z^k = z-\sum_{k=2}^{\infty}\psi(k)a_k z^k, \tag{18}$$

where

$$\psi(k)=\frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}, \quad (k\geq 2), \tag{19}$$

Then

$$0<\psi(k)<\psi(2)=\frac{2}{2+\lambda}. \tag{20}$$

In view of Lemma 3.1, we have

$$(2-\alpha)\delta(n,2)\sum_{k=2}^{\infty}a_k = \sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \leq 1-\alpha, \tag{21}$$

which give

$$\sum_{k=2}^{\infty}a_k \leq \frac{1-\alpha}{(2-\alpha)\delta(n,2)}=\frac{1-\alpha}{(2-\alpha)(n+1)}. \tag{22}$$

Therefore by using (20) and (22) in Equation (18), we get

$$|F(z)|\leq|z|+|\psi(2)||z|^2\sum_{k=2}^{\infty}a_k \leq |z|+\frac{2(1-\alpha)}{(2+\lambda)(2-\alpha)(n+1)}|z|^2 \tag{23}$$

and

$$|F(z)|\geq|z|-|\psi(2)||z|^2\sum_{k=2}^{\infty}a_k \geq |z|-\frac{2(1-\alpha)}{(2+\lambda)(2-\alpha)(n+1)}|z|^2. \tag{24}$$

Thus, the proof of Theorem 3.1 is completed.

Corollary 3.1. Under the hypotheses of Theorem 3.1, $D_z^{-\lambda} f(z)$ is included in a disc with its center at the origin and radius r_1 given by

$$r_1=\frac{1}{\Gamma(2+\lambda)}\left\{1+\frac{(1-\alpha)}{(2+\lambda)(2-\alpha)(n+1)}\right\}. \tag{25}$$

Put $\lambda=0.5, \alpha=\frac{1}{4}$ and $n=0$ in Corollary 3.1 we get the following corollary

Corollary 3.2. under the hypotheses of Theorem 3.1, $D_z^{-\frac{1}{2}} f(z)$ is included in a disc with its center at the origin and radius r_2 given by

$$r_2 = \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left\{1+\frac{\frac{1}{2}}{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}\right\} = 0.852\,. \tag{26}$$

Put $\lambda = \frac{1}{2}$ in Theorem 3.1, we get the following corollary

Corollary 3.3. Let the function $f(z)$ given by (2) be in the class $R_n^*(\alpha)$, then we have

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \le \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1+\frac{2(1-\alpha)}{5(2-\alpha)(n+1)}|z|\right\}, \tag{27}$$

and

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \ge \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1-\frac{2(1-\alpha)}{5(2-\alpha)(n+1)}|z|\right\}. \tag{28}$$

for $0 \le \alpha < 1, n \in N_0$ and $z \in U$. The result is sharp for the function

$$D_z^{-\frac{1}{2}} f(z) = \frac{z^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1-\frac{2(1-\alpha)}{5(2-\alpha)(n+1)}\,z\right\}. \tag{29}$$

Put $n = 0$ in Corollary 3.3, we have the following corollary

Corollary 3.4. Let the function $f(z)$ defined by (2) be in the class $T^*(\alpha)$, then we have

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \le \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1+\frac{2(1-\alpha)}{5(2-\alpha)}|z|\right\}, \tag{30}$$

and

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \ge \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1-\frac{2(1-\alpha)}{5(2-\alpha)}|z|\right\}, \tag{31}$$

for $0 \le \alpha < 1$ and $z \in U$. The result is sharp for the function

$$D_z^{-\frac{1}{2}} f(z) = \frac{z^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1+\frac{2(1-\alpha)}{5(2-\alpha)}z\right\}\,. \tag{32}$$

Put $\alpha = 0$ in Corollary 3.4, we have the following corollary

Corollary 3.5. Let the function $f(z)$ defined by (2) be in the class T^*, then we have

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \le \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1+\frac{1}{5}|z|\right\}, \tag{33}$$

and

$$\left|D_z^{-\frac{1}{2}} f(z)\right| \ge \frac{|z|^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left\{1-\frac{1}{2}|z|\right\}. \tag{34}$$

The result is sharp for the function

$$D_z^{-\frac{1}{2}} f(z) = \frac{z^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \left\{1 - \frac{1}{2} z\right\} = \frac{4}{2.7} z^{\frac{3}{2}} - \frac{2}{2.7} z^{\frac{5}{2}} \tag{35}$$

Now, we Graph the sharp function given by Equation (35) by Complex Tool Program

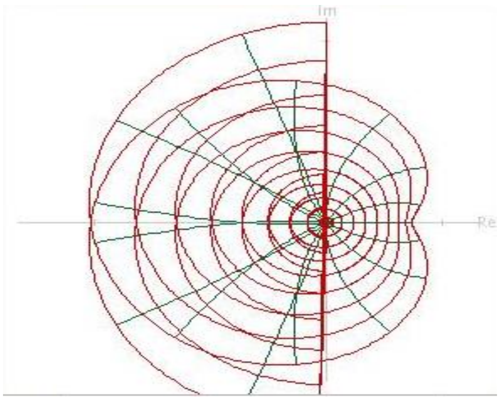

Figure 3.1 the image of unit disc under the function $f(z) = \frac{4}{2.7} z^{\frac{3}{2}} - \frac{2}{2.7} z^{\frac{5}{2}}$

Theorem 3.2. Let the function $f(z)$ defined by (2) be in the class $R_n^*(\alpha)$, then we have

$$\left|D_z^{\lambda} f(z)\right| \le \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)(n+1)} |z|\right\} \tag{36}$$

and

$$\left|D_z^{\lambda} f(z)\right| \ge \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)(n+1)} |z|\right\}, \tag{37}$$

for $0 \le \lambda < 1$ and $z \in U$. The result is sharp for the function $f(z)$ defined by

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)(n+1)} z\right\} \tag{38}$$

Proof. Let

$$G(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+2-\lambda)} a_k z^k = z - \sum_{k=2}^{\infty} k\,\Phi(k) a_k z^k, \tag{39}$$

where

$$\Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+2-\lambda)}, \qquad (k \ge 2), \tag{40}$$

then

$$0<\Phi(k)\le\Phi(2)=\frac{1}{2-\lambda}. \tag{41}$$

In view of Lemma 3.1,

$$\sum_{k=2}^{\infty}k\,a_k\le\frac{1-\alpha}{n+1}+\alpha\sum_{k=2}^{\infty}a_k\le\frac{2(1-\alpha)}{(2-\alpha)(n+1)}. \tag{42}$$

On another hand by using (41) and (42) , we get

$$|G(z)|\le|z|+\Phi(2)|z|^2\sum_{k=2}^{\infty}k\,a_k\le|z|+\frac{2(1-\alpha)}{(2-\alpha)(n+1)(2-\lambda)}|z|^2 \tag{43}$$

and

$$|G(z)|\ge|z|-\Phi(2)|z|^2\sum_{k=2}^{\infty}k\,a_k\ge|z|-\frac{2(1-\alpha)}{(2-\alpha)(n+1)(2-\lambda)}|z|^2. \tag{44}$$

The proof of Theorem 3.2 is complete.

Corollary 3.6. Under the hypotheses of Theorem 3.2, $D_z^{\lambda}f(z)$ is included in a disc with its center at the origin and radius r_2 given by

$$r_2=\frac{1}{\Gamma(2-\lambda)}\left\{1+\frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)(n+1)}\right\}. \tag{45}$$

Put $\lambda=\frac{1}{2}$ in Theorem 3.2, we get the following corollary

Corollary 3.7. Let the function $f(z)$ defined by (2) be in the class $R_n^*(\alpha)$, then

$$\left|D_z^{\frac{1}{2}}f(z)\right|\le\frac{\sqrt{z}}{\frac{1}{2}\Gamma(\frac{1}{2})}\left\{1+\frac{4(1-\alpha)}{3(2-\alpha)(n+1)}|z|\right\} \tag{46}$$

and

$$\left|D_z^{\frac{1}{2}}f(z)\right|\ge\frac{\sqrt{z}}{\frac{1}{2}\Gamma(\frac{1}{2})}\left\{1-\frac{4(1-\beta)}{3(2-\alpha)(n+1)}|z|\right\} \tag{47}$$

for $z\in U$. The result is sharp for the function

$$D_z^{\frac{1}{2}}f(z)=\frac{\sqrt{z}}{\frac{1}{2}\Gamma(\frac{1}{2})}\left\{1-\frac{4(1-\alpha)}{3(2-\alpha)(n+1)}z\right\}. \tag{48}$$

Put $\lambda=0$ in Theorem 3.2, we get the following corollary

Corollary 3.8. Let the function $f(z)$ defined by (2) be in the class $R_n^*(\alpha)$, then

$$|f(z)|\le|z|+\frac{(1-\alpha)}{(2-\alpha)(n+1)}|z|^2 \tag{49}$$

and

$$|f(z)|\ge|z|-\frac{(1-\alpha)}{(2-\alpha)(n+1)}|z|^2 \tag{50}$$

for $0\le\alpha<1$ and $z\in U$. The result is sharp for the function

$$f(z)=z-\frac{(1-\alpha)}{(2-\alpha)(n+1)}z^2. \tag{51}$$

Remark 3.1. The result obtained in Corollary 3.8 give the same result of Hossen [5].
Put $\alpha = 0$ in Corollary 3.9, we get the following corollary

Corollary 3.10. Let the function $f(z)$ defined by (2) be in the class R_n^*, then

$$|f(z)| \le |z| + \frac{1}{2(n+1)}|z|^2 \tag{52}$$

and

$$|f(z)| \ge |z| - \frac{1}{2(n+1)}|z|^2 \tag{53}$$

The result is sharp for the function

$$f(z) = z - \frac{1}{2(n+1)}z^2 \tag{54}$$

Put $n = 0$ in Corollary 3.10, we get the following corollary

Corollary 3.11. Let the function $f(z)$ defined by (2), be in the class T^*, then

$$|f(z)| \le |z| + \frac{1}{2}|z|^2, \tag{55}$$

and

$$|f(z)| \ge |z| - \frac{1}{2}|z|^2 \tag{56}$$

for $z \in U$. The result is sharp for the function

$$f(z) = z - \frac{1}{2}z^2. \tag{57}$$

Graph the sharp function given by equation (57) by Complex Tool Program by figure 3.2.

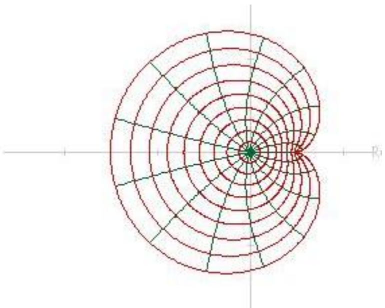

figure 3.2 the image of unit disc under the function $f(z) = z - \frac{1}{2}z^2$

Put $n = 0$ in Theorem 3.2, we get the following corollary

Corollary 3.12. Let the functions $f(z)$ defined by (2) be in the class $S^*(\alpha)$, then

$$\left|D_z^\lambda f(z)\right| \le \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 + \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)}|z|\right\} \tag{58}$$

and

$$\left|D_z^\lambda f(z)\right| \ge \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)}|z|\right\} \tag{59}$$

for $0 \le \lambda < 1$ and $z \in U$. The result is sharp for the function

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{2(1-\alpha)}{(2-\lambda)(2-\alpha)} z \right\} \tag{60}$$

IV. CONCLUSION

This work is generalization for well-known distortion inequality to certain classes of univalent functions which are studied by different authors. In addition, we get number of corollaries by fractional order.

REFERENCES

[1] W. W. Bell, Special Functions for Scientists and Engineers, New Jersey Toronto Melbourne, 1968.
[2] R. H. Buti, A subclass of multivalent functions with negative coefficients defined by integral operator with fractional derivative. I, Gen. Math. Notes, 19(2013), No. 2, 95 – 103.
[3] N. E. Cho and M. K. Aouf, Some application of fractional calculus operators to a certain subclasses of analytic functions with negative coefficients, Tr. J. of Mathematics 20(1996), 553 - 562.
[4] A. W. Goodman, Univalent Functions. I, Mariner Publishing Co. , INC. , 1983.
[5] H. M. Hossen, Generalization of certain classes of univalent functions with negative coefficients, S. U. "BABES BOLYAI" MATHEMATICAL, (2001), No. 2, 41 – 58.
[6] U. N. Katugampola, A new approach to generalized fractional derivatives, Bulletin of Math. Anal. Appl. 6(2014), No. 4, 1 – 15.
[7] Seymour Lipschutz, Murray Spiegel, John Liu, Schaum's Outline of Mathematical Handbook of Formulas and Tables, 4th Edition 2400 Formulas + Tables , Mc-Graw Hill Companies, 2013.
[8] R. Magin, Fractional calculus in bioengineering. I, Crit. Rev. Biomed Eng. 32(2004), No. 1, 1 – 104.
[9] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, Chaos. Soliyons Fractals 7(1996), 1461 – 1477.
[10] A. O. Mostafa and M. K. Aouf, Some subordination results for p-valent functions associated with differintegral operator, Journal of Fractional Calculus and Application, 5(2014), No. 1, 11 - 25.
[11] S. Owa, On the distortion theorems. I, Kyungpook Math. J. 18(1978), 53 – 59.
[12] S. Owa, On new classes of univalent functions with negative coefficients, Bull. Korean Math. Soc. 22(1985), No. 1, 43 – 52.
[13] St. Ruscheweyh, New criteria for univalent functions, Proc. Am. Math. Soc. 49(1975), 109-115.
[14] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1975), 109 – 116.
[15] H. M. Srivastava and S. Owa, An application of the fractional derivative, Math. Japon. 29(1984), 383 – 389.

First Author: H. M. Hossen is Associate Professor in Mathematic Department - Jazan University.
Second Author: N. Asker is Assistant Professor in Mathematic Department - Jazan University.
Third Author: F. H. Majrapi is Student at the Faculty of Science in Mathematic Department - Jazan University.
Four Author: B. M. Al-Awad is Graduate of Faculty of Science in Mathematic Department - Jazan University.